CONTROL SYSTEMS

Mrs. Janani R

Department of Electronics and Instrumentation Engineering

Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya

SYLLABUS

UNIT 1: INTRODUCTION TO CONTROL SYSTEMS

Open Loop and Closed Loop systems: Generalized Block Diagram of a Feedback System: Block diagram algebra – Signal Flow Graph and Mason's Gain Rule. Transfer function models of linear time-invariant systems-Mathematical models of physical systems.

UNIT 2: TIME RESPONSE ANALYSIS

Standard test signals-Time response of first and second order systems for standard test inputs – Steady state error and error constants – Design specifications for second order systems based on the time response. Proportional, Integral and Derivative Controllers.

UNIT 3: STABILITY ANALYSIS

Concept of Stability: Necessary conditions for Stability – BIBO Stability-Routh-Hurwitz Criterion. Root locus concept: Guidelines for sketching root loci – Root locus plot for continuous time systems. Introduction to design – lag, lead and lag-lead compensators in time domain – Root locus method

UNIT 4: FREQUENCY RESPONSE ANALYSIS

Relationship between time and frequency response, Polar plots, Bode plots, Nyquist stability criterion-Relative stability using Nyquist criterion-gain and phase margin. Controller Design specifications in frequency domain – Design of Compensators in frequency domain

UNIT 5: STATE VARIABLE ANALYSIS

Concepts of state variables, State space model, Diagonalization of State Matrix – Solution of state equations – Eigen values and Stability Analysis – Concept of controllability and observability – Pole placement by state feedback – State space models of linear discrete time systems.

REFERENCES:

1) J.Nagarath and M. Gopal, Control Systems Engineering, Fourth Edition, New Age International (P) Ltd., 2009.
2) M.Gopal, Control Systems Principles and Design, McGraw-Hill Education, Fourth Edition, 2012.

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Chapter 1
INTRODUCTION TO CONTROL SYSTEMS

1.1. CONTROL SYSTEMS

Control system theory evolved as an engineering discipline and due to universality of the principles involved, it is extended to various fields like economy, sociology, biology, medicine, etc.

Control theory has played a vital roles in the advance of engineering and science. The automatic control has become an integral part of modern manufacturing and industrial processes. When a number of elements or components are connected in a sequence to perform a specific function, the group thus formed is called ***system***. In a system when the output quantity is controlled by varying the input quantity, the system is called ***control system***. The output quantity is called controlled variable or response and input quantity is called command signal or excitation.

1.1.1. OPEN LOOP SYSTEM

Any physical system which does not automatically correct the variation in its output, is called as *open loop system*, or control system in which the output quantity has no effect upon the input quantity are called open-loop control system. This means that the output is not feedback to the input for correction.

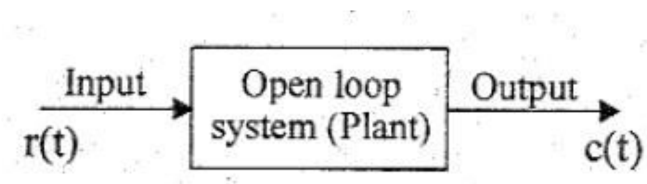


Figure 1-1: Open loop System

In open loop system the output can be varied by varying the input. But due to external disturbances the system output may change. When the output changes due to disturbances, it is not followed by changes in input to correct the output. In open loop systems the changes in output are corrected by changing the input manually.

1.1.2. CLOSED LOOP SYSTEM

Control systems in which the output has an effect upon the input quantity in order to maintain the desired output value are called *closed loop systems*. The open loop system can be modified as closed loop system by providing a feedback. The provision of feedback automatically corrects the changes in output due to disturbances. Hence the closed loop system is also called *automatic control system*. The general block diagram of an automatic control system is shown in Fig. It consists of an error detector, a controller, .and plant and feedback path elements.

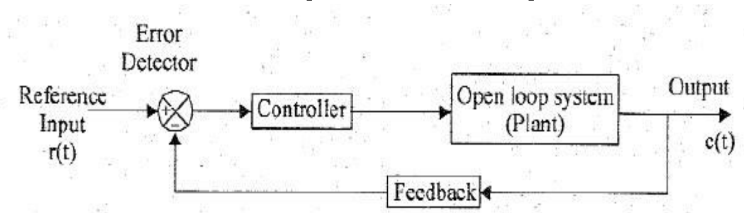


Figure 1-2: Closed loop System

The reference signal corresponds to desired output. The feedback path elements samples the output and converts it to a signal of same type as that of reference signal. The feedback signal is proportional to output signal and it is fed to the error detector. The error signal generated by the error detector is the difference between reference signal and feedback signal. The controller modifies and amplifies the error signal to produce better control action. The modified error signal is fed to the plant to correct its output.

Advantages of open loop systems

- The open loop systems are simple and economical
- The open loop systems are easier to construct
- Generally the open loop systems are stable.

Disadvantages of open loop systems

- The open loop systems are inaccurate and unreliable
- The changes in the output due to external disturbances are not corrected automatically

Advantages of closed loop systems

- The closed loop systems are accurate.
- The closed loop systems are accurate even in the presence of non-linearities.
- The sensitivity of the systems may be made small to make the system more stable.
- The closed loop systems are less affected by noise.

Disadvantages of closed loop systems

- The closed loop systems are complex and costly
- The feedback in closed loop system may lead to oscillatory response.
- The feedback reduces the overall gain of the system.
- Stability is a major problem in closed loop system and more care is needed to design a stable closed loop system

1.2. EXAMPLES OF CONTROL SYSTEMS

Temperature Control System

Open Loop System

The electric furnace shown in fig. is an open loop system. The output in the system is the desired temperature. The temperature of the system is raised by heat generated by the heating element. The output temperature depends on the time during which the supply to heater remains ON. The ON and OFF of the supply is governed by the time setting of the relay. The temperature is measured by a sensor, which gives an analog voltage corresponding to the temperature of the furnace. The analog signal is converted to digital signal by an Analog-to-Digital converter (A/D converter). The digital signal is given to the digital display device to display the temperature. In this system if there is any change in output temperature then the time setting of the relay is not altered automatically.

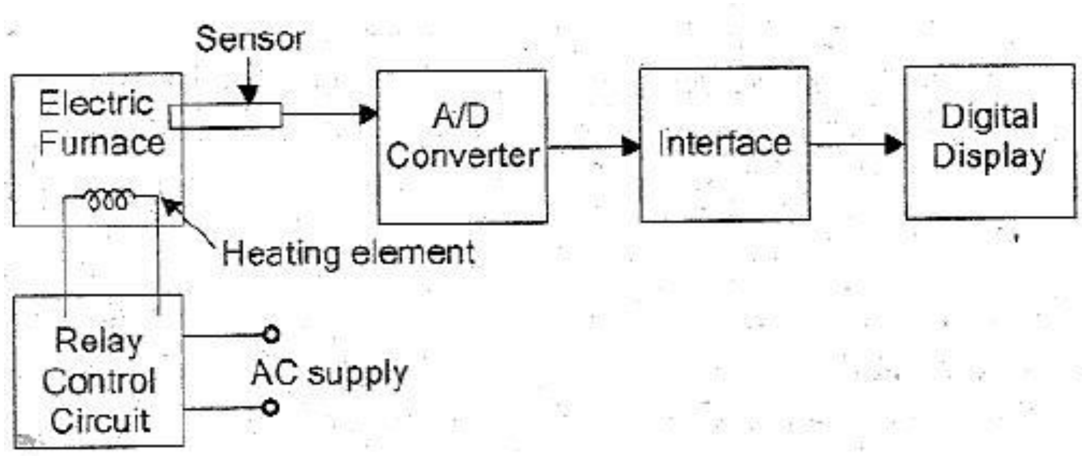


Figure 1-3: Open loop Temperature Control System

Closed Loop System

The electric furnace shown in fig. is a closed loop system. The output of the system is the desired temperature and it depends on the time during which the supply to heater remains ON. The switching ON and OFF of the relay is controlled by a controller which is a digital system or computer. The desired temperature is input to the system through keyboard or as a signal corresponding to desired temperature via ports. The actual temperature is sensed by sensor and converted to digital signal by the A/D converter. The computer reads the actual temperature and compares with desired temperature. If it finds any differences ten it sends signal to switch ON or OFF the relay through D/A converter and amplifier. Thus the system automatically corrects any changes in output. Hence it is a closed loop system.

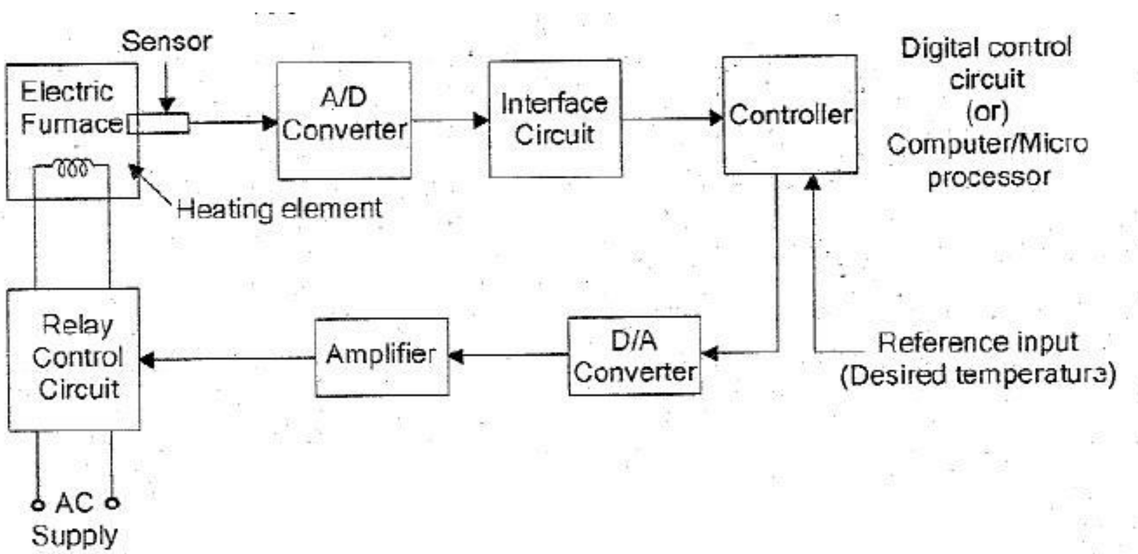


Figure 1-4: Closed loop Temperature Control System

1.3. MATHEMATICAL MODELS OF PHYSCIAL SYSTEMS

A ***control system*** is a collection of physical objects connected together to serve an objective. The input output relations of various physical components of a system are governed by ***differential equations***. The mathematical model of a control system constitutes a set of differential equations. The response or output of the system can be studied by solving the differential equations for various input conditions.

The mathematical model of a system is linear if it obeys the principle of superposition and homogeneity. This principle implies that if a system model has response $y_1(t)$ and $y_2(t)$ to any inputs $X_1(t)$ and $X_2(t)$ respectively, then the system response to the linear combinations of these inputs $a_1X_1(t) + a_2X_2(t)$ is given by linear combination of the individual outputs $a_1y_1(t) + a_2y_2(t)$, where a_1 and a_2 are constants.

A mathematical model will be linear if the differential equations describing the system has constant coefficients. If the coefficients of the differential equation describing the system are constant then the model is ***linear time invariant.*** If the coefficients of differential equations governing the system are functions of time then the model is ***linear time varying.*** The differential equations of a linear time invariant system can be reshaped into different form for the convenience of analysis. One such model for single input and single output system analysis is transfer function of the system. ***The transfer function*** of a system is defined as the ratio of Laplace transform of output to the Laplace transform of input with zero initial conditions.

$$Tansfer\ function = \frac{Laplace\ Transform\ of\ output}{Laplace\ Transform\ of\ input}$$

The transfer function can be obtained by taking Laplace transform of the differential equations governing the system with zero initial conditions and rearranging the resulting algebraic equations to get the ratio of output to input.

1.4. TRANSFER FUNCTION MODELS OF LINEAR TIME INVARIANT SYSTEMS

I. Mechanical Translational Systems

The model of mechanical translational systems can be obtained by using three basic elements ***mass, spring and dash-pot.*** These three elements represents three essential phenomena which occur in various ways in mechanical systems. The weight of the mechanical system is represented by the element ***mass*** and it is assumed to be concentrated at the center of the body.

The elastic deformation of the body can be represented by a ***spring.*** The friction existing in rotating mechanical system can be represented by the ***dash-pot.*** The dash-pot is a piston moving inside a cylinder filled with viscous fluid. When a force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system.

The force acting on a mechanical body are governed by ***Newton's second law of motion.*** For translational systems it states that the sum of forces acting on a body is zero.

LIST OF SYMBOLS USED IN MECHANICAL TRANSLATIONAL SYSTEM

$x = Displacement, m$
$v = \frac{dx}{dt} = Velocity, m/sec$
$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = Acceleration; m/sec^2$
$f = Applied\ force, N\ (Newtons)$
$f_m = Opposing\ force\ offered\ by\ mass\ of\ the\ body, N$

f_k = Opposing force offered by elasticity of the body (spring), N
f_b = Opposing force offered by friction of the body (dash − pot), N
M = Mass, kg
K = Stiffness of spring, N/m
B = Viscous friction co − efficient , N − sec/m

FORCE BALANCE EQUATIONS FOR MECHANICAL ELEMENTS

Consider an ***ideal mass element*** shown in fig. which has negligible friction and elasticity. Let a force be applied on it. The mass will offer an opposing force which is proportional to acceleration of the body.

Let f = Applied force; f_m = Opposing force due to mass

Here $f_m \propto \frac{d^2x}{dt^2}$ or $f_m = M\frac{d^2x}{dt^2}$

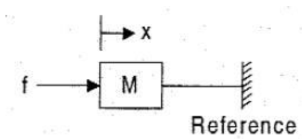
Therefore $f = f_m = M\frac{d^2x}{dt^2}$

Consider an ***ideal frictional element dashpot*** shown in fig. which has negligible mass and elasticity. Let a force be applied on it. The dash-pot will offer an opposing force which is proportional to velocity of the body.

Let f = Applied force; f_b = Opposing force due to friction

Here $f_b \propto \frac{dx}{dt}$ or $f_b = B\frac{dx}{dt}$

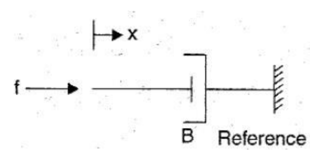
Therefore $f = f_b = B\frac{dx}{dt}$



Ideal mass element.

Consider an ***ideal elastic element spring*** shown in fig. which has negligible mass and friction. Let a force be applied on it. The spring will offer an opposing force which is proportional to displacement of the body.

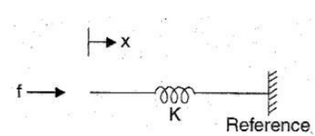
Let f = Applied force; f_k = Opposing force due to elasticity
Here $f_k \propto x$ or $f_k = Kx$
Therefore $f = f_k = Kx$



Ideal dashpot

GUIDELINE TO DETERMINE THE TRANSFER FUNTION OF MECHANICAL SYSTEM

- In mechanical translational system, the differential equations governing the system are obtained by writing force balance equations at nodes in the system.
- The linear displacement of the nodes are assumed as x_1, x_2, x_3, etc., and assign a displacement to each node.
- Draw the free body diagrams of the system. The free body diagram is obtained by drawing which mass separately and then marking all the forces acting on the node. ***Always the opposing force acts in a direction opposite to applied force.*** The mass has to move in the direction of the applied force. Hence the displacement, velocity and acceleration of the mass will be in the direction of the



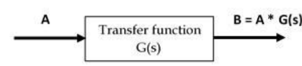
Ideal spring

applied force. If there is no applied force then the displacement, velocity and acceleration of the mass will be in a direction opposite to that of opposing force.

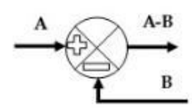
- For each free body diagram, write one differential equation by equating the sum of applied force to the sum of opposing forces.
- Take Laplace transform of differential equations to convert them to algebraic equations. Obtain the ratio between output variable and input variable. This ratio is the transfer function of the system

1.5. BLOCK DIAGRAM ALGEBRA

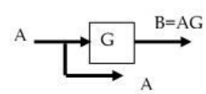
A control system may consist of a number of components. In control engineering to show the functions performed by each component, we commonly use a diagram called the *block diagram.* A ***block diagram*** of a system is a pictorial representation of the functions performed by each component and on the flow of signals. Such a diagram depicts the interrelationships that exist among the various components. The elements of a block diagram are ***block, branch point and summing point.***



BLOCK: In a block diagram all system variables are linked to each other through functional blocks. The ***functional block*** or simply ***block*** is a symbol for the mathematical operation on the input signal to the block that produces the output. The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals. The output signal from the block is given by the product of input signal and transfer function in the block.



SUMMING POINT: ***Summing points*** are used to add two or more signals in the system. Referring to fig, a circle with a cross is the symbol that indicates a summing operation. The plus or minus sign at each arrow indicates whether the signal is to be added or subtracted.



BRANCH POINT: A ***branch point*** is a point from which the signal from a block goes concurrently to other blocks or summing points.

CONSTRUCTING BLOCK DIAGRAM FOR CONTROL SYSTEMS

A control system can be represented diagrammatically by block diagram. The differential equations governing the system are used to construct the block diagram. By taking Laplace transform the differential equations are converted to algebraic equations. The equations will have variables and constants. From the working knowledge of the system the input and output variables are identified and the block diagram for each equation can be drawn. Each equation gives one section of block diagram. The output of one section will be input for another section. The various sections are interconnected to obtain the overall block diagram of the system.

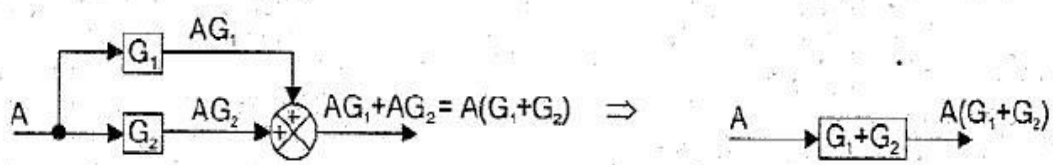
BLOCK DIAGRAM REDUCTION:

The block diagram can be reduced to find the overall transfer function of the system. The following rules can be used for block diagram reduction.

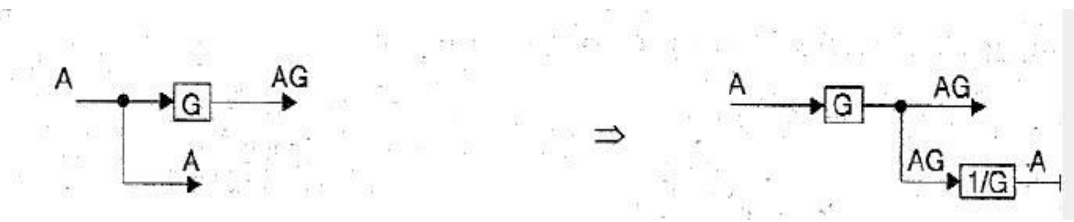
RULES OF BLOCK DIAGRAM ALGEBRA

Rule 1: Combining the blocks in cascade

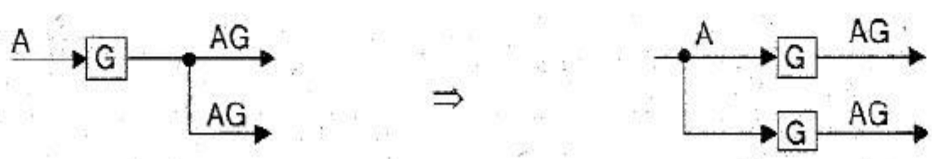
Rule 2: Combining the parallel blocks



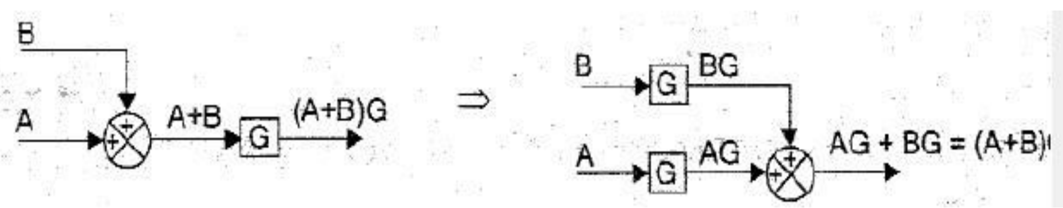
Rule 3: Moving the branch point ahead of the block



Rule 4: Moving the branch point before the block

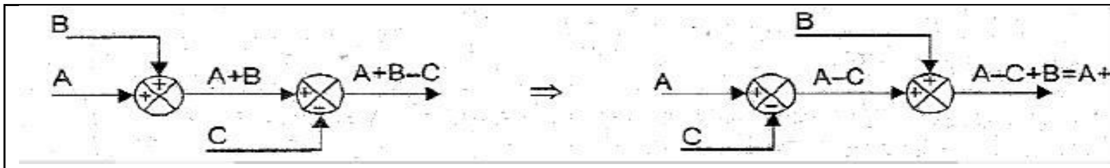


Rule 5: Moving the summing point ahead of the block



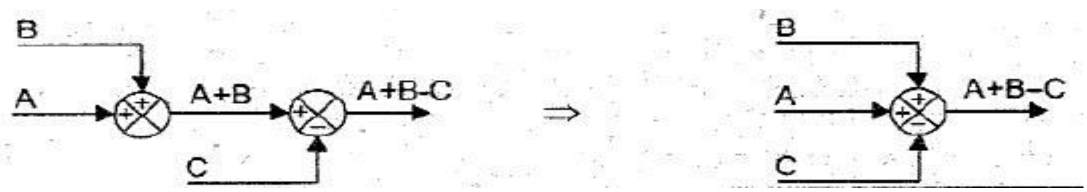
Rule 6: Moving the summing point before the block

Rule 7: Interchanging summing point

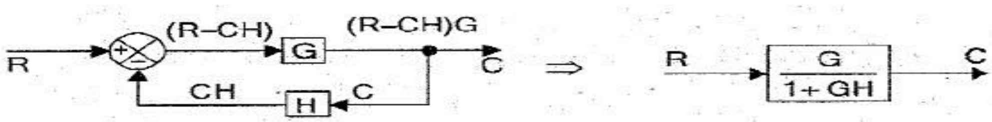


Rule 8: Splitting summing points

Rule 9:Combining summing points



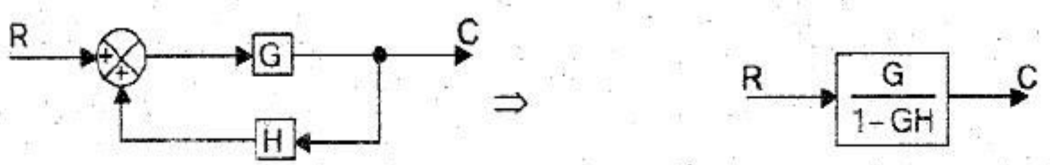
Rule 10: Elimination of (negative) feedback loop



$C = (R - CH)G \Rightarrow C = RG - CHG \Rightarrow C + CHG = RG$

$\therefore C(1 + HG) = RG \Rightarrow \frac{C}{R} = \frac{G}{1+GH}$

Rule 11: Elimination of (positive) feedback loop



1.6. SIGNAL FLOW GRAPH AND MASONS GAIN RULE

The signal flow graph is used to represent the control system graphically and it was developed by SJ Mason. A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking laplace transform, the time domain differential equations governing a control system can be transferred to a set of algebraic equations in s-domain. The signal flow graph of the system can be constructed using these equations. The signal flow graphs depicts the flow of signals from one point of a system to another and gives the relationships among the signals. A signal flow graph consists of a network in which nodes are connected by directed branches. Each node represents a system variable and each branch connected between two nodes acts as a signal multiplier. When a signal pass through a branch, it gets multiplied by the gain of the branch. In a signal flow graph, the signal flows in only one direction. The direction of signal flow is indicated by an arrow placed on the branch and the gain is indicated along the branch.

EXPLANATION OF TERMS USED IN SIGNAL FLOW GRAPH

Node	:	A node is a point representing a variable or signal.
Branch	:	A branch is directed line segment joining two nodes. The arrow on the branch indicates the direction of signal flow and the gain of a branch is the transmittance.
Transmittance	:	The gain acquired by the signal when it travels from one node to another is called transmittance. The transmittance can be real or complex.
Input node (Source)	:	It is a node that has only outgoing branches.
Output node (Sink)	:	It is a node that has only incoming branches
Path	:	A path is a traversal of connected branches in the direction of the branch arrows. The path should not cross a node more than once.
Forward Path	:	It is the path from an input node to an output node that does not cross any node more than once.
Forward path gain	:	It is the product of the branch transmittances of a forward path.
Individual loop	:	It is a closed path starting from a node and after passing through a certain part of a graph arrives at same node without crossing any node more than once.
Loop gain	:	It is the product of the branch transmittances (gains) of a loop.
Non touching Loops	:	If the loops does not have a common node then they are said to be non-touching loops.

PROPERTIES OF SIGNAL FLOW GRAPH

The basic properties of signal flow graph are the following:

i. The algebraic equations which are used to construct signal flow graph must be in the form of cause and effect relationship.
ii. Signal flow graph is applicable to linear systems only.
iii. A node in the signal flow graph represents the variable or signal
iv. A node adds the signals of all incoming branches and transmits the sum to all, outgoing branches.
v. A mixed node which has both incoming and outgoing signals can be treated as an output node by adding an outgoing branch of unity transmittance.
vi. A branch indicates functional dependence of one signal on the other.

vii. The signals travel along branches only in the marked direction and when it travels it gets multiplied by the gain or transmittance of the branch.

viii. The signal flow graph of system is not unique. By rearranging the system equations different types of signal flow graphs can be drawn for a given system.

SIGNAL FLOW GRAPH ALGEBRA

Signal flow graph for a system can be reduced to obtain the transfer function of a system using the following rules. The guideline in developing the rules for signal flow graph algebra is that the signal at node is given by sum of all incoming signals.

Rule 1: Incoming signal to a node through a branch is given by the product of a signal at previous node and the gain of the branch.

Rule 2: Cascade branches can be combined to give a single branch whose transmittance is equal to the product of individual branch transmittance.

Rule 3: Parallel branches may be represented by single branch whose transmittance is the sum of individual branch transmittances.

Rule 4: A mixed node can be eliminated by multiplying the transmittance of outgoing branch (from the mixed node) to the transmittance of all incoming branches to the mixed node.

Rule 5: A loop may be eliminated by writing equations at the input and output node and rearranging the equations to find the ratio of output to input. This ratio gives the gain of resultant branch.

SIGNAL FLOW GRAPH REDUCTION

The signal flow graph of a system can be reduced either by using the rules of a signal flow graph algebra or by using Mason's gain formula. For signal flow graph reduction using the rules of signal flow graph, write equations at every node and then rearrange these equations to get the ratio of output and input (transfer function)

MASON'S GAIN FORMULA:

The Mason's gain formula is used to determine the transfer function of the system from the signal flow graph of the system.

Let R(s) = Input to the system

C(s) = Output of the system

Now, Transfer function of the system, T(s) = C(s)/R(s)

Mason's gain formula states the overall gain of the system [transfer function] as follows,

$$Overall\ gain, T = \frac{1}{\Delta}\sum_{K} P_K \Delta_K$$

Where, T = T(s) = Transfer function of the system

P_K = Forward path gain of K^{th} forward path

K = Number of forward paths in the signal flow graph

Δ = 1 – (Sum of individual loop gains) + (Sum of gain products of all possible combinations of two non-touching loops) – (Sum of gain products of all possible combinations of three non-touching loops) + ……………..

Δ_K = Δ for that part of the graph which is not touching K^{th} forward path.

PROCEDURE FOR CONVERTING BLOCK DIAGRAM TO SIGNAL FLOW GRAPH

The signal flow graph and block diagram of a system provides the same information but there is no standard procedure for reducing the block diagram to find the transfer function of the system. Also the block diagram reducing technique will be tedious and it is difficult to choose the rule to be applied for simplification. The following procedure can be used to convert block diagram to signal flow graph.

1) Assume nodes at input, output, at every summing point, at every branch point and in between cascaded blocks.
2) Draw the nodes separately as small circles and number the circles in the order 1, 2, 3, 4,… etc.
3) From the block diagram find the gain between each node in the main forward path and connect all the corresponding circles by straight line and mark the gain between the nodes.
4) Draw the feed forward paths between various nodes and mark the gain of feed forward path along with sign.
5) Draw the feedback paths between various nodes and mark the gain of feedback paths along with sign.

Chapter 2
TIME RESPONSE ANALYSIS

2.1. TIME RESPONSE

The time response of the system is the output of the closed loop system as a function of time. It is denoted by c(t). The time response can be obtained by solving the differential equation governing the system. Alternatively, the response c(t) can be obtained from the transfer function of the system and the input to the system.

$$The\ closed\ loop\ transfer\ function, \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The output or response in s-domain, C(s) is given by the product of the transfer function and the input R(s). On taking inverse Laplace transform of this product the time domain response, c(t) can be obtained.

$$Response\ in\ s-domain\ C(s) = R(s)M(s)$$

$$Response\ in\ time\ domain\ c(t) = L^{-1}\{C(s)\} = L^{-1}\{R(s) * M(s)\}$$

The time response of a control system consists of two parts: *the transient and the steady state response.* The transient response is the response of the system when the input changes from one state to another. The steady state response is the response as time, **t** approaches infinity.

2.2. TEST SIGNALS

In most of the systems the input signals are not known ahead of time and also it is difficult to express the input signals mathematically by simple equations. The commonly used test input signals are impulse, step, ramp, acceleration and sinusoidal signals.
The standard test signals are,

- Step Signal, Unit Step Signal
- Ramp Signal, Unit Ramp Signal
- Parabolic Signal
- Impulse Signal
- Sinusoidal Signal

STEP SIGNAL

The step signal is a signal whose value changes from zero to A at t = 0 and remains constant at A for t>0. The step signal resembles an actual steady input to a system. A special case of step signal is unit step in which A is unity. The mathematical representation of the step signal is,

$$r(t) = 1 \quad ; \quad t \geq 0 \qquad 1$$

$$= 0 \quad ; \quad t < 0$$

RAMP SIGNAL

The ramp signal is a signal whose value increases linearly with time from an initial value of zero at t=0. The ramp signal resembles a constant velocity input to the system. A special case of step signal is unit ramp signal in which the value of **A** is unity. The mathematical representation of the ramp signal is,

$$r(t) = A\,t \quad ; \quad t \geq 0 \qquad 2$$

$$= 0 \quad ; \quad t < 0$$

PARABOLIC SIGNAL

In parabolic signal, the instantaneous value varies as square of the time from an initial value of zero at t=0. The sketch of the signal with respect to time resembles a parabola. The parabolic signal resembles a constant acceleration input to the system. A special case of parabolic signal is unit parabolic signal in which **A** is unity. The mathematical representation of the parabolic signal is,

$$r(t) = \frac{At^2}{2} \quad ; \quad t \geq 0$$
$$= 0 \quad ; \quad t < 0 \qquad 3$$

IMPULSE SIGNAL

A signal of very large magnitude which is available for very short duration is called ***impulse signal.*** The unit impulse signal is a special case, in which **A** is unity. The impulse signal is denoted by δ(t) and mathematically it is expressed as,

$$\delta(t) = \infty \quad ; \quad t = 0$$
$$= 0 \quad ; \quad t \neq 0 \qquad 4$$

2.3. ORDER OF THE SYSTEM

The input and output relationship of a control system can be expressed by nth order differential equation shown in equation

$$a_0 \frac{d^n}{dt^n} p(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} p(t) + \cdots + a_{n-1} \frac{d}{dt} p(t) + a_n p(t) = b_0 \frac{d^m}{dt^m} p(t) + b_1 \frac{d^{m-1}}{dt^{m-1}} q(t) + \cdots + b_{n-1} \frac{d}{dt} q(t) + b_m q(t) \qquad t \geq 0 \qquad 5$$

Where p(t) = Output Response and q(t) = Input Excitation

The order of the system is given by the order of the differential equation governing the system. If the system is governed by nth order differential equation, then the system is called ***nth Order system.***

Alternatively, the order can be determined from the transfer function of the system. The transfer function of the system can be obtained by taking Laplace transform of the differential equation governing the system and rearranging them as a ratio of two polynomials in s, as given by equation

$$T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s^1 + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s^1 + a_n} \qquad 6$$

Where P(s)= Numerator polynomial and Q(s) = Denominator polynomial

The order of the system is given by the maximum power of **s** in the denominator polynomial, Q(s).

The numerator and denominator polynomial of equation can be expressed in the factorized form as shown

$$T(s) = \frac{P(s)}{Q(s)} = \frac{(s + z_1)(s + z_2) \ldots (s + z_m)}{(s + p_1)(s + p_2) \ldots (s + p_n)} \qquad 7$$

Where $z_1, z_2, \ldots, z_n$ are zeros of the system.

$p_1, p_2, \ldots, p_n$ are poles of the system.

2.4. RESPONSE OF FIRST ORDER SYSTEM FOR UNIT STEP INPUT

The closed loop order system with unity feedback sis shown in

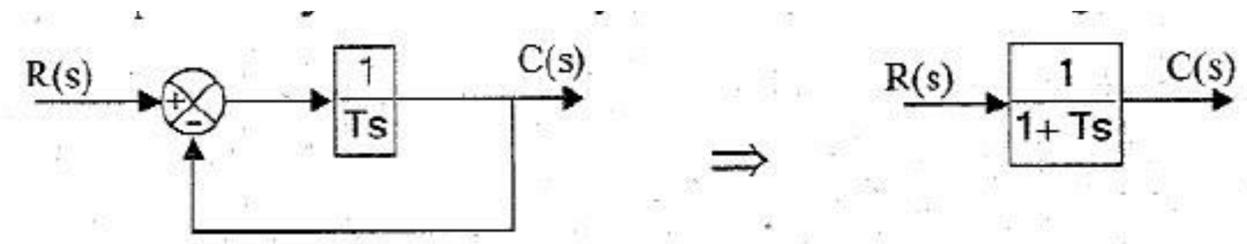


Figure 2-1: Closed loop for first order system

The closed loop transfer function of first order system, $\frac{C(s)}{R(s)} = \frac{1}{1+Ts}$

If the input is unit step then $r(t) = 1, R(s) = \frac{1}{s}$

The response in s-domain $C(s) = R(s)\frac{1}{(1+Ts)} = \frac{1}{s}\frac{1}{(1+Ts)} = \frac{1}{sT(\frac{1}{T}+s)} = \frac{\frac{1}{T}}{s(s+\frac{1}{T})}$

By partial fraction expansion,

$$C(s) = \frac{\frac{1}{T}}{s(s+\frac{1}{T})} = \frac{1}{T} = \frac{A}{s} + \frac{B}{(s+\frac{1}{T})}$$

On solving

$$C(s) = \frac{1}{s} - \frac{1}{(s+\frac{1}{T})}$$

The response in time domain is given by,

$$c(t) = L^{-1}\{C(s)\} = L^{-1}\left\{\frac{1}{s} - \frac{1}{(s+\frac{1}{T})}\right\} = 1 - e^{-\frac{t}{T}}$$ 8

When,	t = 0,	c(t) = 1-e^{0}=1
When,	t = 1T,	c(t) = 1-e^{-1}=0.632
When,	t = 2T,	c(t) = 1-e^{-2}=0.865
When,	t = 3T,	c(t) = 1-e^{-3}=0.95
When,	t = 4T,	c(t) = 1-e^{-4}=0.9817
When,	t = 5T,	c(t) = 1-e^{-5}=0.993
When,	t = ∞,	c(t) = 1-$e^{-\infty}$=1

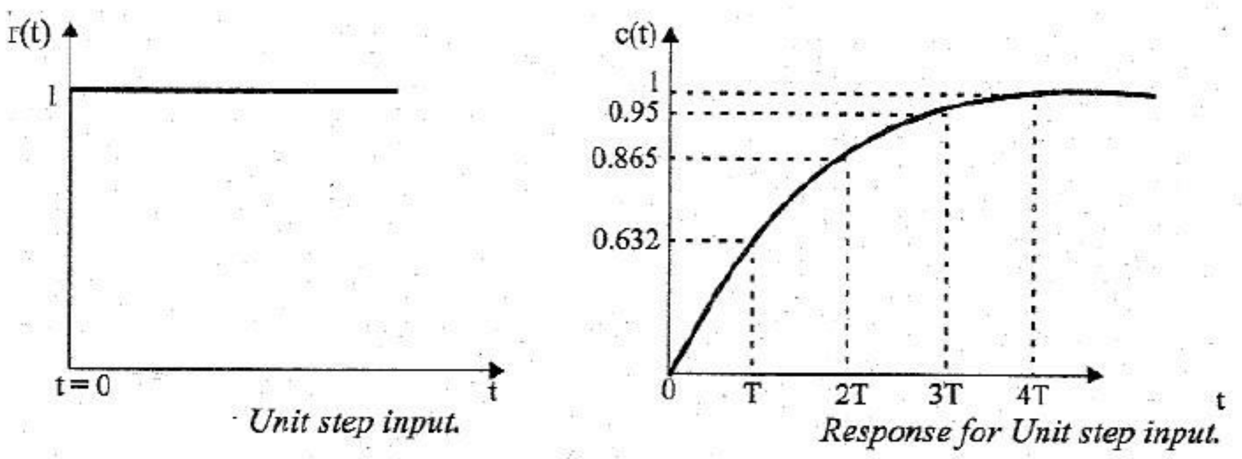
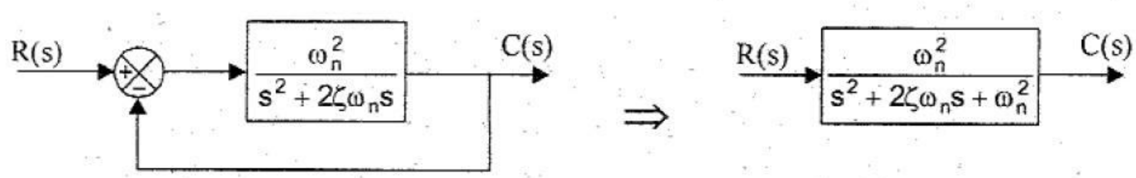


Figure 2-2: Response of First Order System to Unit Step Input

2.5. RESPONSE OF SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The closed loop second order system is shown in figure.



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad 9$$

Where $\omega_n = Undamped\ natural\ frequency, rad/sec$

$\zeta = Damping\ ratio.$

The damping ratio is defined as the ratio of the actual damping to the critical damping. The response c(t) of second order system depends in the value of damping ratio. Depending on the value of ζ the system can be classified into the following four cases

Case 1: Undamped System, $\zeta = 0$

Case 2: Under damped System, $0 < \zeta < 1$

Case 3: Critical damped System, $\zeta = 1$

Case 4: Overdamped System, $\zeta > 1$

RESPONSE OF UNDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad 10$$

For undamped system $\zeta = 0$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2} \qquad 11$$

When the input is unit step $R(s) = 1/s$.

$$Unit\ step\ response, C(t) = 1 - \cos\omega_n t \qquad 12$$

$$Step\ response\ C(t) = A(1 - \cos\omega_n t)$$

RESPONSE OF UNDERDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad 13$$

For undamped system $0 < \zeta < 1$ the roots of the denominator are complex conjugate

When the input is unit step $R(s) = 1/s$.

$$Unit\ step\ response = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\ ;\ \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} \qquad 14$$

$$Step\ response = A(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\ ;\ \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta})$$

RESPONSE OF CRITICALLY DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad 15$$

For critically damped system $\zeta = 1$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} \qquad 16$$

When the input is unit step $R(s) = 1/s$.

$$Unit\ step\ response = 1 - e^{-\omega_n t}(1 + \omega_n t) \qquad 17$$

$$Step\ response = A[1 - e^{-\omega_n t}(1 + \omega_n t)]$$

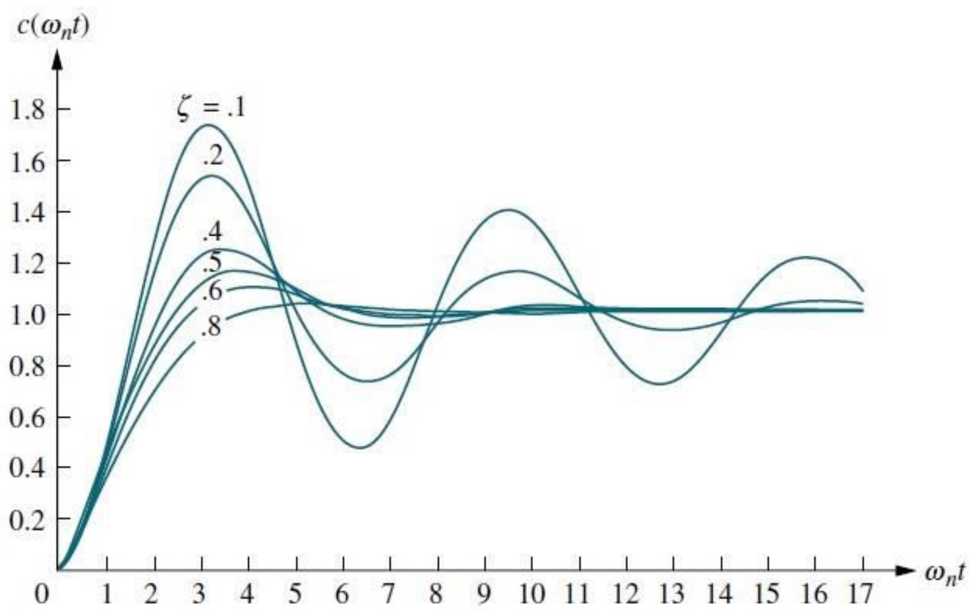


Figure 2-3: Second order underdamped responses for damping ratio values

2.6. TIME DOMAIN SPECIFICATIONS

The desired performance characteristics of control systems are specified in terms of time domain specifications. The transient response of a system to a unit step input depends on the initial conditions. Therefore to compare the time response of various systems it is necessary to start with standard initial conditions. The transient response characteristics of control system to a unit step input is specified in terms of the following time domain specifications.

1. Delay Time t_d
2. Rise Time t_r
3. Peak Time t_p
4. Maximum Overshoot M_p
5. Settling Time t_s

DELAY TIME t_d:

It is the time taken for the response to reach 50% of the final value, for the very first time

RISE TIME tr:

It is the time taken for response to raise from 0 to 100% for the very first time. For underdamped system, the rise time is calculated from 0% to 100%. But for overdamped system, the rise time is calculated as the time taken by the system response to raise from 10% to 90%

$$t_r = \frac{\pi - \theta}{\omega_d}$$ 18

Where $\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$

PEAK TIME tp:

It is the time taken for the response to reach the peak value the very first time.

$$t_p = \frac{\pi}{\omega_d} \quad 19$$

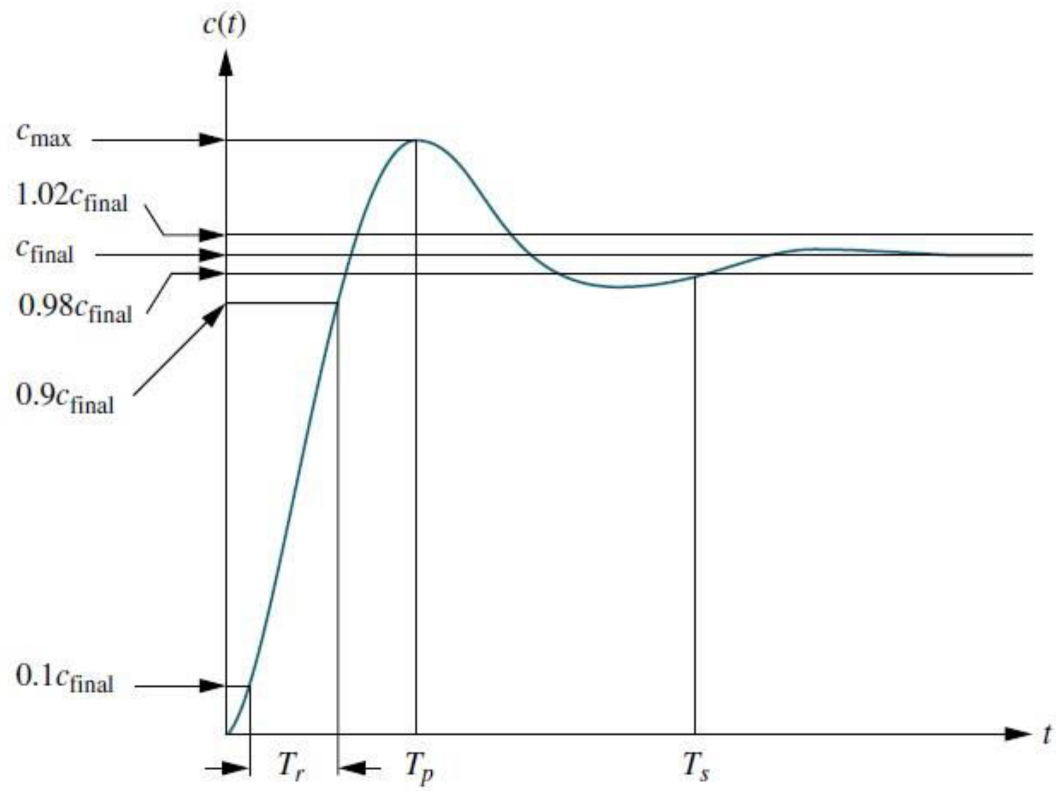


Figure 2-4: Second order system Time Domain Specifications

PEAK OVERSHOOT Mp:

It is defined as the ratio of the maximum peak value to the final value, where the maximum peak value is measured from final value.

$$\%M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} * 100 \quad 20$$

SETTLING TIME:

It is defined as the time taken by the response to reach and stay within a specified error. The usual tolerable error is 2% or 5% of the final value.

$$t_s = \frac{1}{\zeta\omega_n} \quad for\ 2\%\ error \quad 21$$

$$t_s = \frac{3}{\zeta\omega_n} \quad for\ 2\%\ error \quad 22$$

Given the transfer function G(s) determine Rise Time t_r, Peak Time t_p, Maximum Overshoot M_p , Settling Time t_s

$$G(s) = \frac{100}{S^2 + 15s + 100}$$

2.7. TYPE NUMBER OF CONTROL SYSTEMS

The number of poles of the loop transfer function lying at the origin decides the type number of the system. In general, if N is the number of poles at the origin then the type number is N.

If N =0, then the system is Type – 0 System

If N = 1, then the system is Type – 1 System

If N = 2, then the system is Type – 2 System

If N = 3, then the system is Type – 3 System and so on.

2.8. STEADY STATE ERROR

The steady state error is the value of error signal e(t), when t tends to infinity. The steady state error is a measure of system accuracy. The steady state error

$$e_{ss} = \lim_{t\to\propto} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)H(s)}$$ 23

2.9. STATIC ERROR CONSTANTS

When a control system is excited with standard input signal, the steady state error may be zero, constant or infinity. The value of steady state error depends on the type number and the input signal. Type-0 system will have a constant steady state error when the input is step signal. Type-1 system will have a constant steady state error when the input is ramp signal. Type-2 system will have a constant steady state error when the input is parabolic signal.

$$Positional\ error\ constant, K_p = \lim_{s\to 0} G(s)H(s)$$

$$Velocity\ error\ constant, K_v = \lim_{s\to 0} s\, G(s)H(s)$$

$$Acceleration\ error\ constant, K_a = \lim_{s\to 0} s^2 G(s)H(s)$$

For a unity feedback control system the open loop transfer function is $G(s) = \frac{10(S+2)}{S^2(S+1)}$.
Find a) The position, velocity and acceleration error constants

b) The steady state error when the input is R(s) where $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

2.10. RESPONSE WITH P, PI, PD AND PID CONTROLLERS

EFFECT OF PROPORTIONAL CONTROLLER (P - Controller)

The proportional controller produces an output signal which is proportional to error signal. The proportional controller amplifies the error signal and increases the loop gain of the system. The following aspects of system behavior are improved by increasing loop gain.

1) Steady state tracking accuracy.
2) Disturbance signal rejection.
3) Relative stability.

The drawback in proportional control action is that it produces a constant steady state error.

EFFECT OF PROPORTIONAL INTEGRAL CONTROLLER (PI - Controller)

The proportional plus integral controller (PI-controller) produces an output signal consisting of two terms: *one proportional to error signal and the other proportional to the integral of error signal*.

1) PI Controller introduces a zero in the system and increases the order by one.
2) Reduces steady state error.

EFFECT OF PROPORTIONAL DERIVATIVE CONTROLLER (PD - Controller)

The proportional plus derivative controller (PD-controller) produces an output signal consisting of two terms: *one proportional to error signal and the other proportional to the derivative of error signal*.

1) PD Controller introduces a zero in the system and increases damping ratio.
2) Increases peak overshoot and reduce the rise time.

EFFECT OF PROPORTIONAL INTEGRAL DERIVATIVE CONTROLLER (PID - Controller)

A suitable combination of the three basic modes: *Proportional, integral and derivative PID* improve all aspects of the system performance.

1) Proportional controller stabilizes the gain but produces steady state error.

2) Integral controller reduces or eliminates the steady state error.
3) Derivative controller reduces the rate of change of error.

Chapter 3
STABILITY ANALYSIS

3.1. CONCEPT OF STABILITY

DEFINITIONS OF STABILITY

The term stability refers to the stable working condition of a control system. Every working system is designed to be stable. In a stable system, the response or output is predictable, finite and stable for a given input. The different definitions of the stability are the following.

1) A system is stable, if its output is bounded (finite) for any bounded (finite) input.
2) A system is asymptotically stable, if in the absence of the input, the output tends towards zero irrespective of initial conditions.
3) The system is table if for a bounded disturbing input signal the output vanishes ultimately as t approaches infinity.

The following are three points may be stated regarding the stability of the system depending on the location of roots of characteristic equation.

1) If all the roots of characteristic equation has negative real parts, then the system is stable.
2) If any root of the characteristic equation has a positive real part or if there is a repeated root on the imaginary axis then the system is unstable.
3) If the condition (1) is satisfied except for the presence of one or more non-repeated roots on the imaginary axis, then the system is limitedly or marginally stable.

The following conclusions can be made about coefficients of characteristic polynomial.

1) If all the coefficients are positive and if no coefficient is zero, then all the roots are in the left half of s-plane.
2) If any coefficient a_i is equal to zero then, some of the roots may be on the imaginary axis or on the right half of s-plane.
3) If any coefficient a_i is negative then atleast one root is in the right half of s-plane.

Thus the *necessary condition for stability of the system is that all the coefficients of its characteristic polynomial be positive.* If any coefficient is zero/negative, we can immediately say that the system is unstable. When all the coefficients are positive, the system may or may not be stable, because there may be roots in the right half plane and/or on the imaginary axis.

Concept of Stability: Necessary conditions for Stability – BIBO Stability-Routh-Hurwitz Criterion. Root locus concept: Guidelines for sketching root loci – Root locus plot for continuous time systems. Introduction to design – lag, lead and lag-lead compensators in time domain – Root locus method

3.2. ROUTH HURWITZ CRITERION

The Routh-Hurwitz stability criterion is an analytical procedure for determining whether all the roots of a polynomial have negative real part or not.

The first step in analyzing the stability of a system is to examine its characteristic equation. The necessary condition for stability is that all the coefficients of the polynomial be positive. If some of the coefficients are zero or negative it can be concluded that the system is not stable.

When all the coefficients are positive, the system is not necessarily stable. Even though the coefficient are positive, some of the roots may lie on the right half of s-plane or on the imaginary axis. In order for all the roots to have negative real parts, it is necessary but not sufficient that all coefficients of the characteristic equation be positive. If all the coefficients of the characteristic equation are positive, then the system may be stable and one should proceed further to examine the sufficient conditions of stability. The Routh stability criterion is based on ordering the coefficients of the characteristic equation, into a schedule called the Routh array as shown below.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s^1 + a_n = 0, where\ a_0 > 0$$

s^n	a_0	a_2	a_4	a_6	a_8	….
s^{n-1}	a_1	a_3	a_5	a_7	a_9	….
s^{n-2}	b_0	b_1	b_2	b_3	b_4	….
s^{n-3}	c_0	c_1	c_2	c_3	c_4	….
s^1	g_0					
s^0	h_0					

The Routh stability criterion can be stated as follows.

"The necessary and sufficient condition for stability is that all of the elements in the first column of the Routh array be positive. If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of the Routh array corresponds to the number of roots of the characteristic equation in the right half of the s-plane".

If the order of sign of first column element is +, +, -, + and +. Then + to − considered as one sign change and – to + as another sign change.

Using Routh Criterion, determine the stability of the system represented by the characteristic equation, $S^4 + 8S^3 + 18S^2 + 16S + 5 = 0$. Comment on the location of the roots of the characteristic equation

Construct Routh array and determine the stability of the system whose characteristic equation is, $S^6 + 2S^5 + 8S^4 + 12S^3 + 20S^2 + 16S + 16 = 0$. Also determine the number of roots lying on right half of s-plane, left half of s-plane and on imaginary axis.

By Routh stability Criterion, determine the stability of the system represented by the characteristic equation, $9S^5 - 20S^4 + 10S^3 - S^2 - 9S - 10 = 0$. Comment on the location of the roots of the characteristic equation

3.3. ROOT LOCUS CONCEPT

The root locus technique is a powerful tool for adjusting the location of closed loop poles to achieve the desired system performance by varying one or more system parameters. The path taken by the roots of characteristic equation when open loop gain K is varied from 0 to ∞ are called ***root loci***.

CONSTRUCTION OF ROOT LOCUS:

The exact root locus is sketched by trial and error procedure. A set of rules have been developed to reduce the task involved in sketching root locus and to develop a quick approximate sketch. From the approximate sketch, a more accurate root locus can be obtained by a few trials.

RULES FOR CONSTRUCTION OF ROOT LOCUS:

RULE .1. The root locus is symmetrical about the real axis.

RULE .2. Each branch of the root locus originates from an open-loop pole corresponding to K =0 and terminates at either on a finite open loop zero corresponding to K = ∞. The number of branches of the root locus terminating on infinity is equal to n-m.

RULE .3. Segments of the real axis having an odd number of real axis open-loop poles plus zeros to their right are parts of the root locus.

RULE .4. The n-m root locus branches that tend to infinity, do so along straight line asymptotes making angles with the real axis given by $\phi_A = \frac{180(2q+1)}{n-m}$; $q = 0, 1, 2, 3, \dots, n-m$

RULE .5. The point of intersection of the asymptotes with the real axis is at $s = \sigma_A$ where,

$$\sigma_A = \frac{Sum\ of\ Poles - Sum\ of\ Zeros}{n-m}$$

RULE .6. The breakaway and breakin points of the root locus are determined from the roots of the equation dK/ds = 0.

RULE .7. The angle of departure from a complex open-loop pole is given by,

$$\phi_P = \pm 180(2q+1) + \phi \,;\, q = 0, 1, 2, ...$$

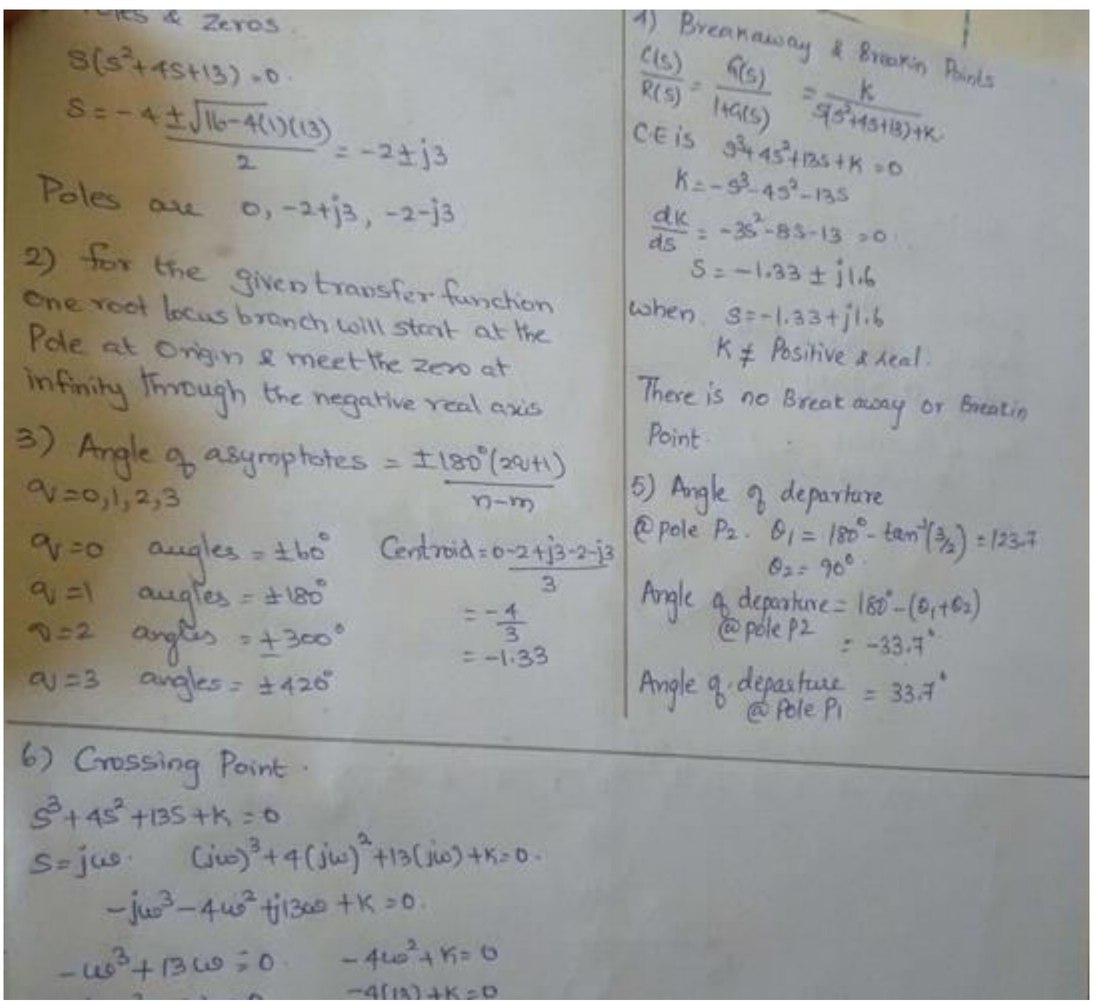
The angle of arrival from a complex open-loop zero is given by,

$$\phi_z = \pm 180(2q+1) + \phi \,;\, q = 0, 1, 2, ...$$

RULE .8. The points of intersection of root locus branches with the imaginary axis can be determined by use of the Routh criterion.

A unity feedback control system has an open loop transfer function,

$G(s) = \frac{K}{S(S^2+4S+13)}$. Sketch the root locus.



Poles & Zeros

$S(S^2+4S+13) = 0$

$S = \frac{-4 \pm \sqrt{16-4(1)(13)}}{2} = -2 \pm j3$

Poles are $0, -2+j3, -2-j3$

2) For the given transfer function one root locus branch will start at the pole at origin & meet the zero at infinity through the negative real axis

3) Angle of asymptotes $= \frac{\pm 180^\circ(2q+1)}{n-m}$

$q = 0,1,2,3$

$q=0$ angles $= \pm 60^\circ$

$q=1$ angles $= \pm 180^\circ$

$q=2$ angles $= \pm 300^\circ$

$q=3$ angles $= \pm 420^\circ$

Centroid $= \frac{0-2+j3-2-j3}{3} = -\frac{4}{3} = -1.33$

4) Breakaway & Breakin Points

$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{S(S^2+4S+13)+K}$

C.E is $S^3+4S^2+13S+K = 0$

$K = -S^3-4S^2-13S$

$\frac{dK}{ds} = -3S^2-8S-13 = 0$

$S = -1.33 \pm j1.6$

when $S = -1.33+j1.6$

$K \neq$ Positive & Real.

There is no Break away or Breakin Point.

5) Angle of departure

@ pole P_2. $\theta_1 = 180^\circ - \tan^{-1}(3/2) = 123.7$

$\theta_2 = 90^\circ$

Angle of departure @ pole P2 $= 180^\circ - (\theta_1+\theta_2) = -33.7^\circ$

Angle of departure @ Pole P1 $= 33.7^\circ$

6) Crossing Point

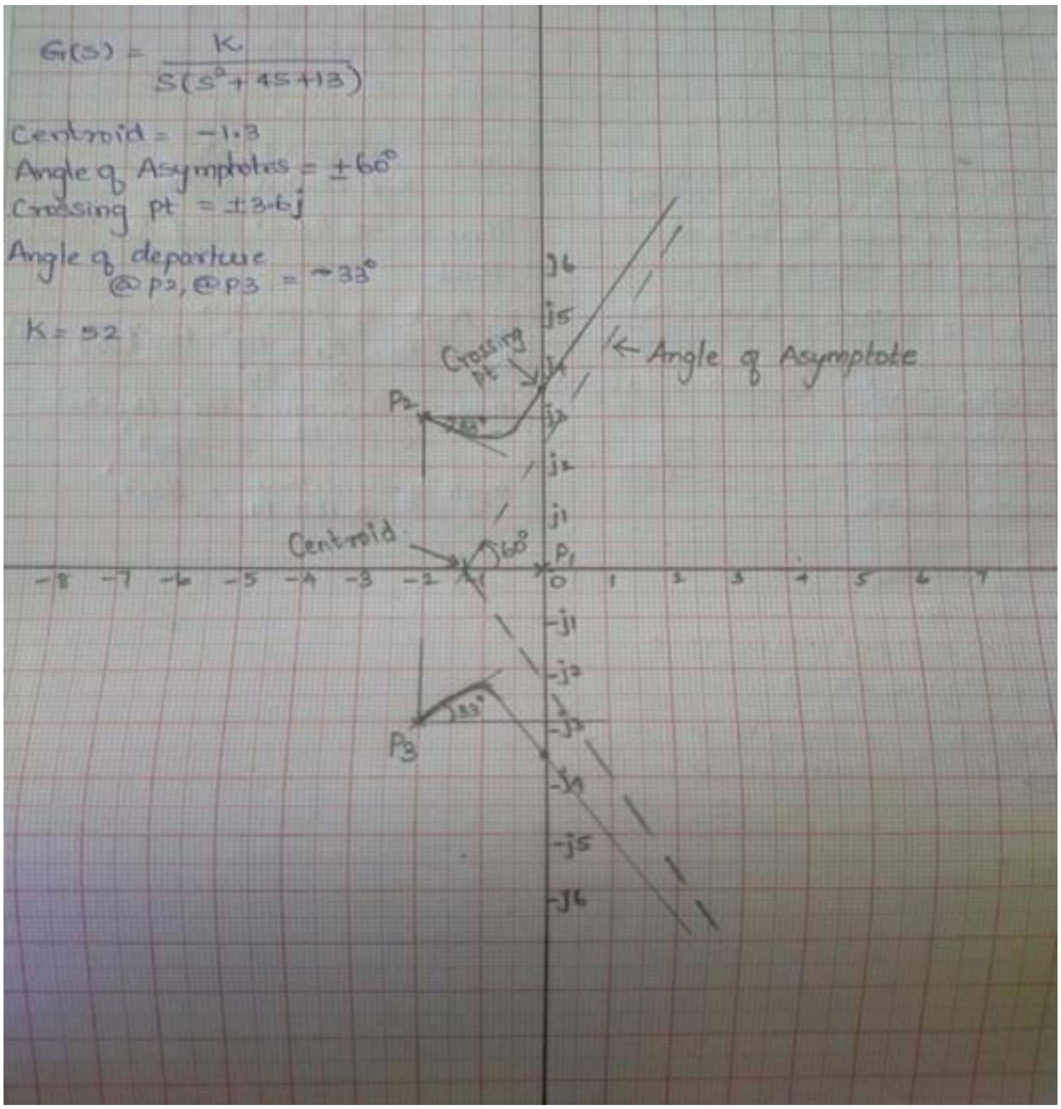
$S^3+4S^2+13S+K = 0$

$S = j\omega$. $(j\omega)^3+4(j\omega)^2+13(j\omega)+K = 0$

$-j\omega^3 - 4\omega^2 + j13\omega + K = 0$

$-\omega^3 + 13\omega = 0$ $\quad -4\omega^2 + K = 0$

$-4(13)+K=0$


G(s) = K / s(s² + 4s + 13)
Centroid = -1.3
Angle of Asymptotes = ±60°
Crossing pt = ±3.6j
Angle of departure @ P2, @ P3 = ~33°
K = 52
Crossing pt
Angle of Asymptote
Centroid
60°
33°
P1
P2
P3
j6
j5
j4
j3
j2
j1
-j1
-j2
-j3
-j4
-j5
-j6

Sketch the root locus of the system whose open loop transfer function, $G(s) = \frac{K}{S(s+2)(s+4)}$

3.4. COMPENSATOR DESIGN

The choice between series compensation and parallel compensation depends on

1) Nature of signals in the system.
2) Power levels at various points
3) Components available
4) Designers experience
5) Economic consideration and so on.

In control systems, compensation is required in the following situations,

1) *When the system is absolutely unstable, then compensation is required to stabilize the system and also to meet the desired performance.*
2) *When the system is stable, compensation is provided to obtain the desired performance.*

PROCEDURE FOR DESIGN OF LAG COMPENSATOR USING ROOT LOCUS

Step. 1 Draw the root locus of uncompensated system.

Step. 2 Determine the dominant pole, s_d. Draw a straight line through the origin with an angle $\cos^{-1}\zeta$ with respect to negative real axis. The intersection point of the straight line with root locus gives the dominant pole s_d

Step. 3 Determine the open loop gain of the uncompensated system at s= s_d. Let this gain be K. The open loop gain K at s = s_d on root locus is given by

$$K = \frac{Product\ of\ vector\ lengths\ from\ s_d\ to\ open\ loop\ poles}{Product\ of\ vector\ lengths\ from\ s_d\ to\ open\ loop\ zeros}$$

Step. 4 Calculate the parameter, β of the compensator.

Step. 5 Determine the transfer function of lag compensator. The zero of the lag compensator (1/T) is chosen to be 10% of the second pole of uncompensated system.

$$Transfer\ function\ of\ lag\ compensator\ G_c(s) = \beta\frac{(1 + sT)}{(1 + s\beta T)}$$

Step. 6 Determine the open loop transfer function of the compensated system. The lag compensator is connected in series with the plant.

Step. 7 Check whether the compensated system satisfies the steady state error requirement. If it is satisfied, then the design is accepted otherwise repeat the design by modifying the location of poles and zeros of the compensator.

The forward path transfer function of a certain unity feedback control system is given by $(s) = \frac{K}{S(S+2)(S+8)}$. Design a suitable lag compensator so that the system meets the following specifications (i) Percentage overshoot ≤ 16% for unit step input. (ii) Steady state error ≤ 0.125 for unit ramp input.

Chapter 4
FREQUENCY RESPONSE ANALYSIS

4.1. FREQUENCY RESPONSE

The frequency domain transfer function $T(j\omega)$ is a complex function of ω. Hence it can be separated into magnitude function and phase function. Now, the magnitude and phase functions will be real functions of ω, and they are called ***frequency response***

4.2. FREQUENCY DOMAIN SPECIFICATIONS

The performance and characteristics of a system in frequency domain are measured in terms of frequency domain specifications. The requirements of a system to be designed are usually specified in terms these specifications. The frequency domain specifications are

Resonant Peak (Mr)

The maximum value of the magnitude of closed loop transfer function is called the resonant peak Mr. A large resonant peak corresponds to a large overshoot in transient response.

Resonant Frequency (ω_r)

The frequency at which the resonant peak occurs is called resonant frequency ω_r. This is related to the frequency of oscillation in the step response and thus it is indicative of the speed of transient response.

Bandwidth (ω_b)

The bandwidth is the range of frequencies for which the system normalized gain is more than -3db. The frequency at which the gain is -3db is called cut-off frequency. Bandwidth is usually defined for closed loop system and it transmits the signals whose frequencies are less than the cut-off frequency. The bandwidth is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics and rise time.

Gain Margin (K_g)

The gain margin Kg is given by the reciprocal of the magnitude of open loop transfer function at phase cross over frequency. The frequency at which the phase of open loop transfer function is 180° is called *phase cross over frequency* ω_{pc}

Phase Margin (γ)

The phase margin is defined as the additional phase lag to be added at the gain cross over frequency in order to bring the system to the verge of instability. It is obtained as given below

$$\text{Phase Margin } \gamma = 180° + \varphi_{gc}$$

4.3. FREQUENCY RESPONSE PLOTS

BODE PLOT

The bode plot is a frequency response plot of the sinusoidal transfer function of a system. It consists of two graphs, one is a plot of the magnitude of a sinusoidal transfer function versus log ω. The other is a plot of the phase angle of a sinusoidal transfer function versus log ω.

> Sketch Bode plot for the following transfer function and determine the system gain K for the gain cross over frequency to be 5 rad/sec.
>
> $$G(s) = \frac{Ks^2}{(1 + 0.2s)(1 + 0.02s)}$$

$$G(s) = \frac{Ks^2}{(1+0.2s)(1+0.02s)}$$

1) $s = j\omega$. $K = 1$

$$G(j\omega) = \frac{K(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)} \qquad G(j\omega) = \frac{(j\omega)^2}{(1+j0.2\omega)(1+j0.02\omega)}$$

2) Corner frequencies. $\omega_{c1} = \frac{1}{0.2} = 5$ rad/sec $\qquad \omega_{c2} = \frac{1}{0.02} = 50$ rad/sec.

3) Magnitude plot.

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$(j\omega)^2$	–	40	–
$\frac{1}{1+0.2j\omega}$	5	–20	40 – 20 = 20
$\frac{1}{1+0.02j\omega}$	50	– 20	20 – 20 = 0

4) Choose $\omega_l = 0.5$ rad/sec $\qquad \omega_h = 100$ rad/sec.

At $\omega = \omega_l \Rightarrow A = 20\log|(j\omega)|^2 = 20\log(\omega^2) = -12$db (–40)

$\omega = \omega_{c1} \quad A = 20\log|(j\omega)^2| = 20\log(\omega^2) = 28$db (0)

$\omega = \omega_{c2} \quad A = \text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}} + A_{(at\ \omega=\omega_{c1})} = 20\times\log\frac{50}{5} + 28 = 48$db (20)

$\omega = \omega_h \quad A = 0\times\log\frac{100}{50} + 48 = 48$db. (20)

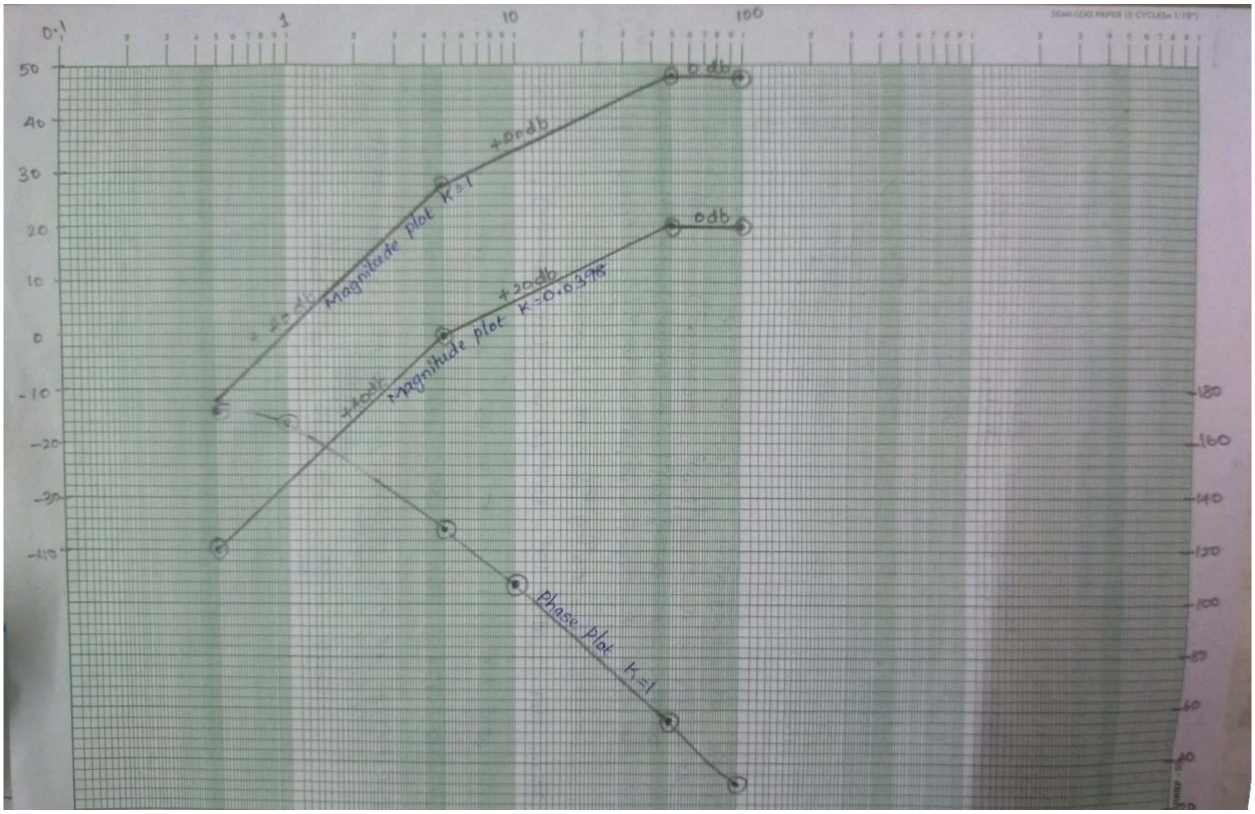
5) Phase plot. $\phi = 180^\circ - \tan^{-1}(0.2\omega) - \tan^{-1}(0.02\omega)$

ω	0.5	1	5	10	50	100
φ	174	168	130	106	50	30

6) CALCULATION OF K :- @ given $\omega = 5$ rad/sec $\Rightarrow$ gain = 28db

$20\log K = -28$db

$\log K = -\frac{28}{20} \qquad K = 10^{\left(-\frac{28}{20}\right)} = 0.0398.$



Examples to practice

Sketch Bode plot for the following transfer function and determine phase margin and gain margin

$$G(s) = \frac{75(1 + 0.2s)}{s(s^2 + 16s + 100)}$$

Sketch Bode plot for the following transfer function and obtain the gain and phase cross over frequencies.

$$G(s) = \frac{10}{s(1 + 0.4s)(1 + 0.1s)}$$

POLAR PLOT

The polar plot of a sinusoidal transfer function G(jω) is a plot of the magnitude of G(jω) versus the phase angle of G(jω) on polar coordinates as ω is varied from zero to infinity. The polar plot is usually plotted on a polar graph sheet.

The open loop transfer function of a unity feedback system is given by.

$$G(s) = \frac{1}{s(1+s)(1+2s)}$$

Sketch the polar plot and determine the gain margin and phase margin.

The open loop transfer function of a unity feedback system is given by.

$$G(s) = \frac{1}{s(1+s)(1+2s)}$$

Sketch the polar plot and determine the gain margin and phase margin.

1) $G(s) = \frac{1}{s(1+s)(1+2s)}$

a) $s = j\omega$ $G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+2j\omega)}$

b) $\omega_{c_1} = 0.5$ rad/sec $\omega_{c_2} = 1$ rad/sec

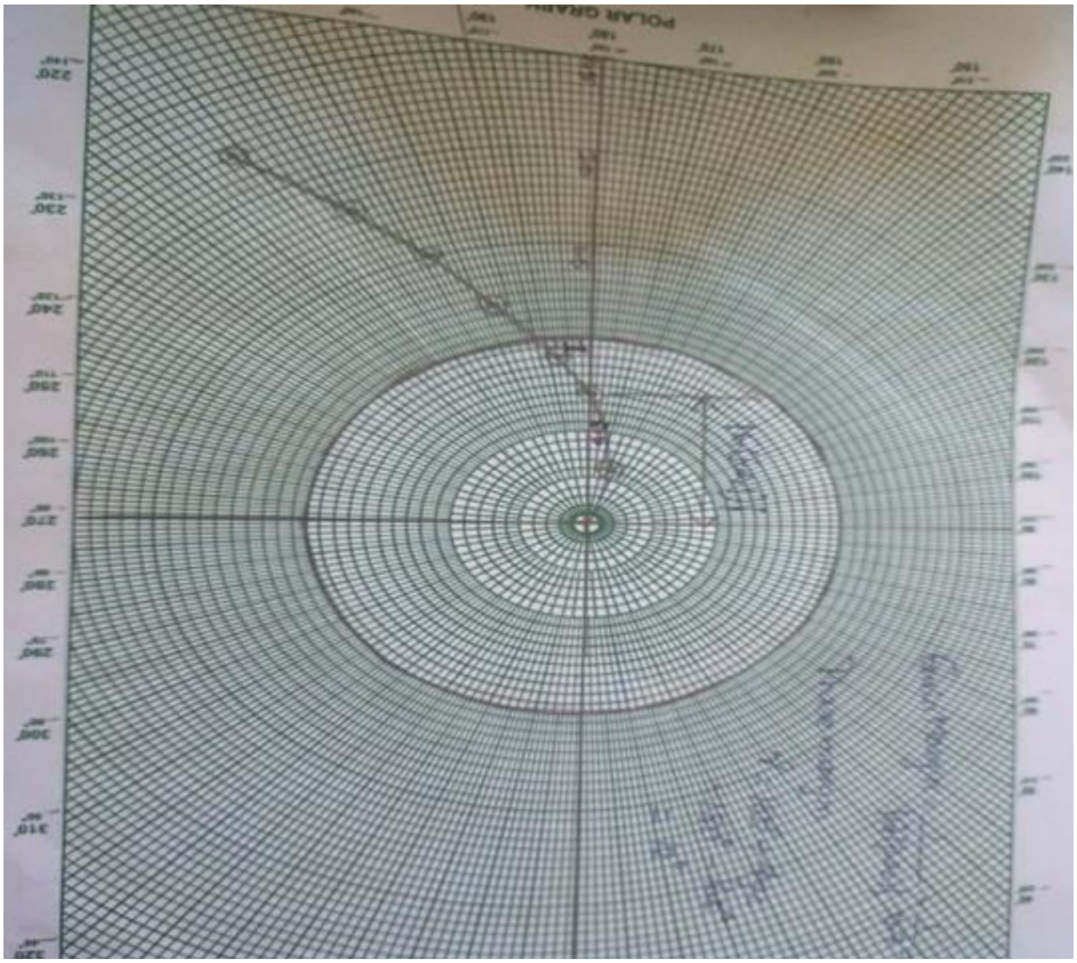
c) $|G(j\omega)| = \frac{1}{\omega\sqrt{(1+\omega^2)(1+4\omega^2)}} = \frac{1}{\omega\sqrt{1+5\omega^2+4\omega^4}}$

d) $\angle G(j\omega) = -90^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$

ω rad/sec	0.35	0.4	0.45	0.5	0.6	0.7	1.0
|G(jω)|	2.2	1.8	1.5	1.2	0.9	0.7	0.3
∠G(jω)	-144	-150	-156	-162	-171	-180	-198

e) Gain margin $K_g = \frac{1}{|G(j\omega_{pc})|} = \frac{1}{0.7} = 1.4286$

Phase margin $\gamma = 180^\circ + \phi_{gc} = 180^\circ - 168^\circ = 12^\circ$



The open loop transfer function of a unity feedback system is given by.

$$G(s) = \frac{1}{s^2(1+s)(1+2s)}$$

Sketch the polar plot and determine the gain margin and phase margin.

4.4. COMPENSATOR DESIGN

When a set of specifications are given for a system, a suitable compensator should be designed so that the overall system will meet the given specification. There are different types of compensator used namely, ***Lag compensator, Lead Compensator and Lag-Lead Compensator.***

Compensation is required in the following situations

- When the system is absolutely unstable, then compensation is required to stabilize the system and also to meet the desired performance.
- When the system is stable, compensation is provided to obtain the desired performance.

The systems with type number 2 and above are usually absolutely unstable systems. Hence for systems with type number 2 and above, lead compensation is required, because the lead compensator increase the margin of stability.

In systems with type number 1 or 0, stability is achieved by adjusting the gain. In such cases any of the three compensators-lag, lead and lag-lead may be used to obtain the desired performance.

The open loop transfer function of a unity feedback system is given by.

$$G(s) = \frac{K}{s(s+4)(s+80)}$$

It is desired to have the phase margin to be atleast 33° and the velocity error constant K_v= 30 sec^{-1}. Design a phase lag series compensator.

The lead compensation increase the bandwidth, which improves the speed of response and also reduces the amount of overshoot. Lead compensation appreciably improves the transient response, whereas there is a small change in steady state accuracy. Lead compensation is provided to make an unstable system as a stable system.

The open loop transfer function of a unity feedback system is given by.

$$G(s) = \frac{K}{s(s+1)(s+5)}$$

It is desired to have the velocity error constant $K_v \geq 50$ and phase margin is $\geq 20°$ Design a suitable lead compensator.

Chapter 5
STATE VARIABLE ANALYSIS

5.1. CONCEPTS OF STATE VARIABLES

The state variable analysis can be applied for any type of systems. The analysis can be carried with initial conditions and can be carried on multiple input and multiple output systems. In this method of analysis, it is not necessary that the state variables represent physical quantities of the system, but variables that do not represent physical quantities and those that are neither measurable not observable may be chosen as state variables.

A set of variables which describes the system at any time instant are called **state variables.** A system consists of m-inputs, p-outputs and n-state variables. The state space representation of the system may be

State variables = $x_1(t), x_2(t), x_3(t), \ldots\ldots, x_n(t)$

Input variables = $u_1(t), u_2(t), u_3(t), \ldots\ldots, u_m(t)$

Output variables = $y_1(t), y_2(t), y_3(t), \ldots\ldots, y_p(t)$

The different variables may be represented by the vectors (column matrix) as shown below.

$$Input\ vector\ U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \quad Output\ vector\ Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_P(t) \end{bmatrix} \quad State\ Variable\ vector\ X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

The state model of a system consists of state equation and output equation. The state equation and output equation together called as state model of the system. Hence the state model of a linear time invariant system (LTI) system is given by the following equations.

$$\dot{X}(t) = A\,X(t) + B\,U(t)$$

$$Y(t) = C\,X(t) + D\,U(t)$$

5.2. STATE DIAGRAM

The pictorial representation of the state model of the system is called ***State diagram.*** The state diagram of the system can be either in Block Diagram form or in Signal flow graph form. The state diagram describes the relationships among the state variables and provides physical interpretations of the state variables. The time domain state diagram may be obtained directly from the differential equation governing the system and this diagram can be used for simulation of the system in analog computers.

The state diagram of a state model is constructed using three basic elements ***Scalar, Adder and Integrator.***

Scalar: The scalar is used to multiply a signal by a constant. The input signal x(t) is multiplied by the scalar **a x(t).**

Adder: The adder is used to add two or more signals. The output of the adder is the sum of incoming signals.

Integrator: The integrator is used to integrate the signal. They are used to integrate the derivatives of state variables to get the state variables. The initial conditions of the state variable can be added by using an adder after integrator.

Obtain the state model of the system whose transfer function is given as

$$\frac{Y(s)}{U(s)} = \frac{10}{S^3 + 4S^2 + 2S + 1}$$

Solution

Given that,

$$\frac{Y(s)}{U(s)} = \frac{10}{S^3 + 4S^2 + 2S + 1}$$

$$Y(s)[S^3 + 4S^2 + 2S + 1] = 10U(s)$$

$$Y(s)S^3 + 4S^2Y(s) + 2SY(s) + Y(s) = 10U(s)$$

On taking inverse Laplace transform of the equation, we get,

$$\dddot{y} + 4\ddot{y} + 2\dot{y} + y = 10u$$

Let us define state variables as follows,

$$x_1 = y\,; x_2 = \dot{y}\,;\ x_3 = \ddot{y}$$

$$\dddot{y} = \dot{x_3}\ \ \dot{y} = x_2\,; \ddot{y} = x_3\, and\ y = x_1$$

$$\dot{x_3}\ + 4x_3 + 2x_2 + x_1 = 10u$$

$$\dot{x_3} = -4x_3 - 2x_2 - x_1 + 10u$$

The state equations are $\dot{x_1} = x_2\,; \dot{x_2} = x_3\,; \dot{x_3} = -x_1 - 2x_2 - 4x_3 + 10u$

The output equation is $y = x_1$

The state model in the matrix form is

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} [u]$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

5.3. EIGEN VALUES AND EIGEN VECTORS

A nonzero column vector X is an eigenvector of a square matrix A, if there exists a scalar λ such that AX = λX, then λ is eigen value of A. Eigen value may be zero but the corresponding vector may not be a zero vector.

Find f(A) = A^7 for A = $\begin{bmatrix} 0 & 3 \\ -2 & -5 \end{bmatrix}$

$$[\lambda I - A] = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} \lambda & -3 \\ 2 & \lambda + 5 \end{bmatrix}$$

$$The\ characteristic\ equation\ is\ given\ by\ (\lambda + 2)(\lambda + 3) = 0$$

The eigen values λ_1, λ_2 are roots of characteristic equation

$$\lambda_1 = -2, \lambda_2 = -3$$

Given that f(A) = A^7, f(λ) = λ^7,

When $\lambda_1 = -2$; f(-2) = $(-2)^7$ = -128

When $\lambda_1 = -3$; f(-3) = $(-3)^7$ = -2187

$$f(\lambda_i) = \alpha_0 + \alpha_1 \lambda_i$$

When $\lambda_1 = -2$;

$$-128 = \alpha_0 - 2\alpha_1$$

When $\lambda_1 = -3$;

$$-2187 = \alpha_0 - 3\alpha_1$$

On solving $\alpha_0 = 3990$ and $\alpha_1 = 2059$ $f(A) = \alpha_0 I + \alpha_1 A$

$$A^7 = \begin{bmatrix} 3990 & 6177 \\ -4118 & -6305 \end{bmatrix}$$

Alternate Method:

A = $\begin{bmatrix} 0 & 3 \\ -2 & -5 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} -6 & -15 \\ 10 & 19 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 30 & 57 \\ -38 & -65 \end{bmatrix}$$

$$A^7 = \begin{bmatrix} 3990 & 6177 \\ -4118 & -6305 \end{bmatrix}$$

5.4. CONCEPTS OF CONTROLLABILITY AND OBSERVABILITY

Controllability

A system is said to be completely state controllable if it is possible to transfer the system state from any initial state $X(t_0)$ to any other desired state $X(t_d)$ in specified finite time by a control vector $U(t)$. Consider a system with state equation $\dot{X} = AX + BU$. For this system, a composite matrix

$$Q_c = [B \quad AB \quad A^2B \ldots\ldots \; A^{n-1}B]$$

Where n is the order of the system (n is also equal to number of state variables). The rank of the matrix is n, if the determinant of (nxn) composite matrix Q_c is non zero, then rank of $Q_c = n$ and the system is completely state controllable,

Observability

A system is said to be completely state observable if every state X(t) can be completely identified by measurements of the output Y(t) over a finite time interval. Consider a system with state equation $\dot{X} = AX + BU$. For this system, a composite matrix

$$Q_o = [C^T \quad A^TC^T \quad (A^T)^2C^T \ldots\ldots (A^T)^{n-1}C^T]$$

Where n is the order of the system (n is also equal to number of state variables). In this case, the system is completely observable if the rank of composite matrix Q_o is non-zero.

5.5. CONTROL SYSTEM DESIGN VIA POLE PLACEMENT BY STATE FEEDBACK

In control system design by pole placement or pole assignment technique, the state variables are used for feedback, to achieve desired closed loop poles. The advantage in this system is that the closed loop poles may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix K. The necessary and sufficient condition to be satisfied by the system for arbitrary pole placement is that the system be completely state controllable.

DETERMINATION OF STATE FEEDBACK GAIN MATRIX, K

Consider a linear System described by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{10}{s(s+1)(s+2)}$$

Design a feedback Controller with a State feedback so that the closed loop poles are placed at $-2, -1 \pm j1$

1) Determine the State equation.

$$\frac{Y(s)}{U(s)} = \frac{10}{s(s+1)(s+2)}.$$

$$Y(s)\left[s(s+1)(s+2)\right] = 10\,U(s)$$

$$Y(s)\left[s^3+3s^2+2s\right] = 10\,U(s)$$

$$s^3Y(s) + 3s^2Y(s) + 2sY(s) = 10\,U(s)$$

$$\dddot{y} + 3\ddot{y} + 2\dot{y} = 10u.$$

Consider $x_1 = y$

$x_2 = \dot{y}$

$x_3 = \ddot{y}$

$$\dot{x}_3 + 3x_3 + 2x_2 = 10u$$

$$\dot{x}_3 = -2x_2 - 3x_3 + 10u$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_1 = x_2$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 10\end{bmatrix}u$$

2) Check for Controllability.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & 6 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ -30 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & 6 & 7 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ -30 \\ 70 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 10 & -30 \\ 10 & -30 & 70 \end{bmatrix}$$

$$|Q_c| = 10(0-100) = -1000$$

$|Q_c| \neq 0$ System is Controllable.

$$Q_c^{-1} = \frac{1}{|Q_c|}\begin{bmatrix} -200 & -300 & -100 \\ 300 & -100 & 0 \\ -100 & 0 & 0 \end{bmatrix}^T$$

$$Q_c^{-1} = \begin{bmatrix} -0.20 & 0.3 & 0.1 \\ -0.3 & 0.1 & 0 \\ 0.1 & 0 & 0 \end{bmatrix}$$

3) Find desired characteristic Polynomial.

$\mu_1 = -2 \quad \mu_2 = -1+j1$ and $\mu_3 = -1-j1$

$$(\lambda-\mu_1)(\lambda-\mu_2)(\lambda-\mu_3) = (\lambda+2)(\lambda+1-j1)(\lambda+1+j1)$$
$$= (\lambda+2)((\lambda+1)^2-(j1)^2)$$
$$= (\lambda+2)\left[\lambda^2+2\lambda+1+1\right]$$
$$= (\lambda+2)(\lambda^2+2\lambda+2)$$
$$= \lambda^3+2\lambda^2+2\lambda+2\lambda^2+4\lambda+4$$
$$= \lambda^3+4\lambda^2+6\lambda+4 \quad \text{— (A)}$$

4) State feedback Matrix, K

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$K = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}.$$

$$|\lambda I-(A-BK)| = 0 = |\lambda I - A + BK| = 0$$

$$= \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 10K_1 & 10K_2 & 10K_3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 10K_1 & 10K_2 & 10K_3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 10K_1 & 2+10K_2 & \lambda+3+10K_3 \end{bmatrix}$$

$$|\lambda I - A+BK| = \lambda\left(\lambda^2+3\lambda+10K_3\lambda+2+10K_2\right)+1\left(10K_1\right)$$
$$= \lambda^3+(3+10K_3)\lambda^2+(2+10K_2)\lambda+10K_1 \quad \text{— (B)}$$

Comparing (A) and (B)

$3 + 10K_3 = 4.$ $\quad 2 + 10K_2 = 6$ $\quad 10K_1 = 4.$

$10K_3 = 1$ $\quad 10K_2 = 4$ $\quad K_1 = 0.4$

$K_3 = 0.1$ $\quad K_2 = 0.4$

The state feedback gain matrix

$$K = \begin{bmatrix} 0.4 & 0.4 & 0.1 \end{bmatrix}$$

<u>Assignment</u>

en

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 10 \\ 1 \\ 0 \end{bmatrix} u.$$

Determine State feedback gain matrix with Poles at $-1 \pm j2, -6$.